APPLICATION OF AN OPERATIONAL METHOD FOR THE SOLUTION OF A PROBLEM ON THE DEVELOPMENT OF THE FLOW OF A VISCOPLASTIC MEDIUM IN THE INITIAL PORTION OF A CYLINDRICAL TUBE

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A solution is given in general form, with end-effect taken into account, for the problem concerning the motion of a viscoplastic medium.

We consider a viscoplastic medium having the rheological equation

$$\Pi_0 = 2\left(\eta + \frac{\tau_0}{h}\right)\dot{\Phi}. \tag{1}$$

The distribution of velocities at the entrance to a cylindrical tube is given in cylindrical coordinates

$$v_z = \psi(r) \text{ for } z = 0 \tag{2}$$

along with the no-slip condition on the tube wall

$$v_z = 0 \text{ for } r = R, \tag{3}$$

where R is the tube radius. Conditions (2) and (3) serve as boundary conditions for the problem. We assume that

$$v_r = v_\varphi = 0, \tag{4}$$

$$v_z = v_z(r, z) \tag{5}$$

and that the medium is incompressible

$$\dot{\Phi}_0 = \dot{\Phi}.$$

We introduce Eq. (1) into the equation of motion of a continuous medium

$$\operatorname{div}\Pi = \rho(\mathbf{a} - \mathbf{F}) \tag{6}$$

and represent the resulting equation in terms of cylindrical coordinates, taking Eqs. (4), (5) into account and neglecting body forces. As a result we obtain

$$\frac{\partial P}{\partial \varphi} = 0, \tag{7}$$

$$2\left(\eta + \frac{\tau_0}{h}\right)\frac{\partial}{\partial z}e_{rz} - \frac{2\tau_0}{h^2}\frac{\partial h}{\partial z}e_{rz} - \frac{\partial P}{\partial r} = 0, \tag{8}$$

$$2\left(\eta + \frac{\tau_0}{h}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}(re_{rz}) + \frac{1}{r}\frac{\partial}{\partial z}(re_{zz})\right] - \frac{2\tau_0}{h^2}\left(\frac{\partial h}{\partial r}e_{rz} + \frac{\partial h}{\partial z}e_{zz}\right) - \frac{\partial P}{\partial z} = 0, \tag{9}$$

where

$$e_{zz} = \frac{\partial v_z}{\partial z}; \quad e_{rz} = \frac{1}{2}\frac{\partial v_z}{\partial r}.$$

We neglect the product of the derivatives with respect to z by e_{zz}, $(\partial/\partial z)e_{rz}$, and $\partial h/\partial z$. In addition Eq. (8) gives $\partial P/\partial r = 0$, and upon taking account of Eq. (7), we obtain

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$$P = P(z). \tag{10}$$

We transform Eq. (9), assuming that in terms containing e_{rz}, $h = |\partial v_z/\partial r|$, and since $\partial v_z/\partial r < 0$, then $h = -\partial v_z/\partial r$ (no slippage at the walls).

Equation (9) then takes the form

$$\eta \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) - \frac{\tau_0}{r} - \frac{\partial P}{\partial z} + 2\left(\eta + \frac{\tau_0}{h}\right)\frac{\partial}{\partial z} e_{zz} = 0. \tag{11}$$

We consider h to be constant, taken equal to a mean value h_c. Equation (11) then assumes the form

$$\eta \frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) - \frac{\tau_0}{r} + i + 2\left(\eta + \frac{\tau_0}{h_c}\right) = 0,$$

where $i = -\partial P/\partial z$ (piezometric slope) is considered to be constant.

We obtain, finally,

$$2\left(\eta + \frac{\tau_0}{h_c}\right)\frac{\partial^2 v_z}{\partial z^2} + \eta r \frac{\partial^2 v_z}{\partial r^2} + \eta \frac{\partial v_z}{\partial r} = \frac{\tau_0}{r} - i. \tag{12}$$

We introduce the notation

$$\frac{2}{\eta}\left(\eta + \frac{\tau_0}{h_c}\right) = A; \quad \frac{1}{\eta}\left(\frac{\tau_0}{r} - i\right) = \alpha(r). \tag{13}$$

Equation (12) is reducible to the form

$$A \frac{\partial^2 v_z}{\partial z^2} + r \frac{\partial^2 v_z}{\partial r^2} + \frac{\partial v_z}{\partial r} = \alpha(r) \tag{14}$$

with the boundary conditions (2) and (3).

To solve Eq. (14) we apply the Laplace transform

$$F(s) = \int_0^\infty \exp(-sz) f(z)\, dz,$$

which enables us to pass in Eq. (14) from the space of originals f(z) to the space of transformed functions F(s), with the boundary condition for z, namely $v_z(r, 0) = \psi(r)$, now becoming an initial condition, automatically included in the transformed equation.

The transform of Eq. (14), unlike the original Eq. (14), is now an ordinary differential equation.

The transform of the function $v_z(r, z)$ is

$$U(r, s) = \int_0^\infty \exp(-sz)\, v_z(r, z)\, dz$$

or, more concisely,

$$U(r, s) \bullet\!\!-\!\!\circ\, v_z(r, z),$$

and the transform of its derivative

$$s^2 U(r, s) - s v_z(r, 0) - \frac{\partial v_z(r, 0)}{\partial z} \bullet\!\!-\!\!\circ \frac{\partial^2 v_z}{\partial z^2}$$

or

$$s^2 U(r, s) - s\psi(r) \bullet\!\!-\!\!\circ \frac{\partial^2 v_z}{\partial z^2}.$$

Since the operations of integration with respect to z and differentiation with respect to r are commutative, the transform of Eq. (14) assumes the form

$$r \frac{d^2 U}{dr^2} + \frac{dU}{dr} + s^2 A U = \alpha(r) + A s \psi(r). \tag{15}$$

Equation (15) is a particular case of a nonhomogeneous Bessel equation, its solution being representable in terms of cylindrical functions

$$U(r,s)=\gamma_1 J_0(2s\sqrt{Ar})+\gamma_2 N_0(2s\sqrt{Ar})+J_0\int\frac{J_0\sqrt{r}\,\tilde F(r,s)}{s\sqrt{A}\,(J_0N_1-J_1N_0)}\,dr+N_0\int\frac{J_0\sqrt{r}\,\tilde F(r,s)\,dr}{s\sqrt{A}\,(J_1N_0-J_0N_1)}, \tag{16}$$

where γ_1 and γ_2 are arbitrary constants, determinable from the conditions $v_z(R, z) = 0$, i.e., $U(R, s) = 0$, and assignment of the speed of motion of the fluid core, i.e., for

$$r=r_0=\frac{2\tau_0}{i}\qquad v_z=v_0,$$

where v_0 is the given speed. In the transform space this condition assumes the form

$$U(r_0,s)=\int_0^\infty \exp(-sz)\,v_0dz=\frac{v_0}{s},$$

$$\tilde F(r,s)=\alpha(r)+As\psi(r).$$

Thus for determining γ_1 and γ_2 we have the system of equations

$$0=\gamma_1 J_0(2s\sqrt{AR})+\gamma_2 N_0(2s\sqrt{AR})+J_0\int\frac{J_0\sqrt{r}\,\tilde F(r,s)\,dr}{s\sqrt{A}(J_0N_1-J_1N_0)}\Big|_{r=R}+N_0\int\frac{J_0\sqrt{r}\,\tilde F(r,s)\,dr}{s\sqrt{A}(J_1N_0-J_0N_1)}\Big|_{r=R},$$

$$\frac{v_0}{s}=\gamma_1 J_0(2s\sqrt{Ar_0})+\gamma_2 N_0(2s\sqrt{Ar_0})+J_0\int\frac{J_0\sqrt{r}\,\tilde F(r,s)\,dr}{s\sqrt{A}\,(J_0N_1-J_1N_0)}\Big|_{r=r_0}+N_0\int\frac{J_0\sqrt{r}\,\tilde F(r,s)\,dr}{s\sqrt{A}\,(J_1N_0-J_0N_1)}\Big|_{r=r_0},$$

whence

$$\gamma_1=\frac{\begin{vmatrix}-J_0\int_1\big|_{r=R}-N_0\int_2\big|_{r=R} & N_0(2s\sqrt{AR})\\ \frac{v_0}{s}-J_0\int_1\big|_{r=r_0}-N_0\int_2\big|_{r=r_0} & N_0(2s\sqrt{Ar_0})\end{vmatrix}}{\begin{vmatrix}J_0(2s\sqrt{AR}) & N_0(2s\sqrt{AR})\\ J_0(2s\sqrt{Ar_0}) & N_0(2s\sqrt{Ar_0})\end{vmatrix}},$$

$$\gamma_2=\frac{\begin{vmatrix}J_0(2s\sqrt{AR}) & -J_0\int_1\big|_{r=R}-N_0\int_2\big|_{r=R}\\ J_0(2s\sqrt{Ar_0}) & \frac{v_0}{s}-J_0\int_2\big|_{r=r_0}-N_0\int_2\big|_{r=r_0}\end{vmatrix}}{\begin{vmatrix}J_0(2s\sqrt{AR}) & N_0(2s\sqrt{AR})\\ J_0(2s\sqrt{Ar_0}) & N_0(2s\sqrt{Ar_0})\end{vmatrix}}.$$

Finding the original function in the general case is a very involved problem and obtaining the final solution to a specific problem is beyond the scope of this paper.

NOTATION

Π_0	is the stress-deviator tensor;
$\dot\Phi_0$	is the deviator of deformation tensor;
h	is the intensity of deformation rates;
η	is the plastic viscosity;
τ_0	is the limited shear stress;
z, φ, r	are the cylindrical coordinates;
v_z, v_φ, v_r	are the velocity components in cylindrical coordinates;

a	is the acceleration vector;
F	is the force vector;
e_{rz}, $e_{\varphi z}$ . . .	are the components of deformation rate tensor in cylindrical coordinates;
P	is the pressure;
ρ	is the density of the medium;
● – ○	indicates correspondence of functions in the Laplace transformation [4];
J, N	are the cylindrical Bessel and Neumann functions.

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